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# The evolution to localized and front solutions in a non-Lipschitz reaction-diffusion Cauchy problem with trivial initial data

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**Abstract**

In this paper, we establish the existence of spatially inhomogeneous classical self-similar solutions to a non-Lipschitz semi-linear parabolic Cauchy problem with trivial initial data. Specifically we consider bounded solutions to an associated two-dimensional non-Lipschitz non-autonomous dynamical system, for which, we establish the existence of a two-parameter family of homoclinic connections on the origin, and a heteroclinic connection between two equilibrium points. Additionally, we obtain bounds and estimates on the rate of convergence of the homoclinic connections to the origin.

*Keywords:* semi-linear parabolic PDE, self-similar, non-Lipschitz, homoclinic connection, heteroclinic connection
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## 1. Introduction

In this paper, we study classical bounded solutions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ to the non-Lipschitz semi-linear parabolic Cauchy problem

$$u_t - u_{xx} = u|u|^{p-1} \quad \text{on } \mathbb{R} \times (0, T], \tag{1}$$

$$u = 0 \quad \text{on } \mathbb{R} \times \{0\}, \tag{2}$$

with $0 < p < 1$ and $T > 0$ (which we henceforth refer to as [CP]). The primary achievement of the paper is the establishment of the existence of a two-parameter family of localized spatially inhomogeneous solutions to [CP] for which $u(x, t) \to 0$ as $|x| \to \infty$ uniformly for $t \in [0, T]$; the secondary achievement of the paper is the establishment of front solutions to [CP], which approach $\pm(1 - p)^{1/(1-p)}t$ as $|x| \to \pm\infty$ uniformly for $t \in [0, T]$. We note here that for $p \geq 1$ in (1), then the unique bounded classical solution with initial data (2) is the trivial solution, see for example [1, Theorem 4.5].

Qualitative properties of non-negative (non-positive) solutions to (1) when $0 < p < 1$, with non-negative (non-positive) initial data, and for which $u(x, t)$ is bounded as $|x| \to \infty$ uniformly for $t \in [0, T]$, have been determined in [2], [3], [4], [5] and [6]. However, we note that any non-negative (non-positive) classical bounded solution to [CP] must be spatially homogeneous for $t \in [0, T]$, see for example [2, Corollary 2.6]. Thus, the solutions constructed in this paper are two signed on $\mathbb{R} \times [0, T]$. The authors are currently unaware of any studies of two signed solutions to (1)-(2) with $0 < p < 1$. Generic local results for spatial homogeneity of solutions to semi-linear parabolic Cauchy problems with homogeneous initial data depend upon uniqueness results, see for example, [6]. For results concerning the related problem of asymptotic homogeneity (in general, asymptotic symmetry) as $t \to \infty$ of non-negative (non-positive) global solutions to semi-linear parabolic Cauchy problems, we refer the reader to the survey article [7].

Non-negative (non-positive), spatially inhomogeneous solutions to (1) for $p > 1$ have been considered in [8], [9] [10], [11], [12], [13], [14], [15], [16] and [17] with the focus primarily on critical exponents for finite time blow-up of solutions, and conditions for the existence of global solutions (see the review articles [18] and [19]). Moreover,

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for $p > 1$, solutions to (1) with two signed initial data have been considered in [20] and [21], whilst boundary value problems have been studied in [22] and [23].

The paper is structured as follows; in Section 2 we introduce the self-similar solution structure for [CP], and hence, determine an ordinary differential equation related to (1); the remainder of the paper concerns the study of particular solutions to this ordinary differential equation, which is re-written as an equivalent two-dimensional non-autonomous dynamical system. Specifically, in Section 3 we establish the existence of a two-parameter family of homoclinic connections on the equilibrium $(0,0)$. Additionally, we determine bounds and estimates on the asymptotic approach of these solutions to $(0,0)$. In Section 4, we establish the existence of a heteroclinic connection between the equilibrium points $(\pm(1-p)^{1/(1-p)},0)$.

## 2. Self-Similar Structure

With $0 < p < 1$ and $T > 0$, we refer to $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ as a solution to [CP] when $u$ satisfies (1)-(2) with regularity,

$$u \in L^\infty(\mathbb{R} \times [0,T]) \cap C(\mathbb{R} \times [0,T]) \cap C^{2,1}(\mathbb{R} \times (0,T]). \tag{3}$$

Observe that $u^\pm : \mathbb{R} \times [0,T] \to \mathbb{R}$ given by

$$u^\pm(x,t) = \pm((1-p)t)^{1/(1-p)} \quad \forall (x,t) \in \mathbb{R} \times [0,T]$$

are the maximal and minimal solutions to [CP] (see [1, Chapter 8]), and hence any solution $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ to [CP] must satisfy,

$$u^-(x,t) \le u(x,t) \le u^+(x,t) \quad \forall (x,t) \in \mathbb{R} \times [0,T]. \tag{4}$$

To construct spatially inhomogeneous solutions to [CP], we consider, for any fixed $x_0 \in \mathbb{R}$, self-similar solutions $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ of the form,

$$u(x,t) = \begin{cases} w\left(\frac{x-x_0}{t^{1/2}}\right) t^{1/(1-p)} & ,(x,t) \in \mathbb{R} \times (0,T], \\ 0 & ,(x,t) \in \mathbb{R} \times \{0\}, \end{cases} \tag{5}$$

with $w : \mathbb{R} \to \mathbb{R}$ to be determined. Now, $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ given by (5) is a solution to [CP] if and only if there exist constants $\alpha, \beta \in \mathbb{R}$ such that $w : \mathbb{R} \to \mathbb{R}$ satisfies the following zero-value problem, namely,

$$w'' + \frac{1}{2}\eta w' + w|w|^{p-1} - \frac{1}{(1-p)}w = 0 \quad \forall \eta \in \mathbb{R}, \tag{6}$$

$$w(0) = \alpha, \quad w'(0) = \beta, \tag{7}$$

$$w \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{8}$$

Here $\eta = (x-x_0)/t^{1/2}$, and we observe that the ordinary differential equation (6) is both non-autonomous and non-Lipschitz. It is convenient to introduce

$$x = w, \quad y = w',$$

after which the problem (6)-(8) is equivalent to the zero-value problem for the two-dimensional, non-Lipschitz, non-autonomous, dynamical system,

$$x' = y \tag{9}$$

$$y' = \frac{1}{(1-p)}x - x|x|^{p-1} - \frac{1}{2}\eta y \quad \forall \eta \in \mathbb{R}, \tag{10}$$

$$(x(0), y(0)) = (\alpha,\beta), \tag{11}$$

$$(x,y) \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{12}$$

We refer to the equivalent zero-value problems in (6)-(8) and (9)-(12) as (S). Our objective is now to investigate those $(\alpha,\beta) \in \mathbb{R}^2$ for which (S) has a non-trivial solution. It is instructive to note, at this stage, via (4), that we may conclude that any solution to (S) must satisfy the inequality,

$$-(1-p)^{\frac{1}{(1-p)}} \leq w(\eta) \leq (1-p)^{\frac{1}{(1-p)}} \quad \forall \eta \in \mathbb{R}, \tag{13}$$

whilst, following [2, Corollary 2.6], any non-constant solution to (S) must be two-signed in $w$.

## 3. Homoclinic Connections

In this section we establish the existence of a two parameter family of homoclinic connections for (S) on the equilibrium point $(0,0)$ of the dynamical system (9)-(10), and establish decay rates to the equilibrium point $(0,0)$ as $|\eta| \to \infty$ on these homoclinic connections.

### *3.1. Existence*

In this subsection, we establish the existence of homoclinic connections attached to the equilibrium point $(x,y) = (0,0)$ of the dynamical system (9)-(10). To begin, observe that $\mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^2$, where

$$\mathbf{Q}(x,y,\eta) = (Q_1, Q_2)(x,y,\eta) = \left(y, \frac{1}{(1-p)}x - x|x|^{p-1} - \frac{1}{2}\eta y\right) \quad \forall (x,y,\eta) \in \mathbb{R}^3 \tag{14}$$

is such that $\mathbf{Q} \in C(\mathbb{R}^3)$, but also that $\mathbf{Q}$ is not locally Lipschitz continuous on $\mathbb{R}^3$ (note that $\mathbf{Q}$ is locally Lipschitz continuous on $\mathbb{R}^3 \backslash N$, with $N$ any neighbourhood of the plane $x = 0$). We now have,

**Theorem 1.** *The problem (S) with zero-value $(\alpha,\beta) \in \mathbb{R}^2$ has a solution for $\eta \in [-\delta, \delta]$ (not necessarily unique), where $\delta = 1/(1+M)$ and*

$$M = \max_{(x,y,\eta)\in R} |\mathbf{Q}(x,y,\eta)|$$

*with*

$$R = \{(x,y,\eta) \in \mathbb{R}^3 : |x-\alpha| \leq 1,\ |y-\beta| \leq 1,\ |\eta| \leq 1\}.$$

*Proof.* This follows immediately from the Cauchy-Peano Local Existence Theorem (see [24, Chapter 1, Theorem 1.2]) since $\mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^2$ is such that $\mathbf{Q} \in C(\mathbb{R}^3)$. $\square$

**Remark 1.** When $\alpha \neq 0$, then the solution to (S) with zero-value $(\alpha,\beta) \in \mathbb{R}^2$ is unique for $\eta \in [-\delta', \delta']$ for some $0 < \delta' \leq \delta$. In addition, the problem (S) with zero-value $(\pm(1-p)^{1/(1-p)}, 0)$ has the unique global solution

$$(x(\eta), y(\eta)) = (\pm(1-p)^{1/(1-p)}, 0) \quad \forall \eta \in \mathbb{R}. \tag{15}$$

This follows since $\mathbf{Q}$ is locally Lipschitz in a neighbourhood of $(\pm(1-p)^{1/(1-p)}, 0)$ respectively. Also, the problem (S) with zero-value (0,0) has the unique global solution,

$$(x(\eta), y(\eta)) = (0,0) \quad \forall \eta \in \mathbb{R}.$$

In this case uniqueness does not follows immediately, since $\mathbf{Q}$ is not locally Lipschitz continuous in any neighborhood of $(0,0)$, but instead follows after further qualitative results have been established for solutions to (S) (see Remark 2).

We now introduce the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$V(x,y) = \frac{1}{2}y^2 - \frac{1}{2(1-p)}x^2 + \frac{1}{(1+p)}|x|^{1+p} \quad \forall (x,y) \in \mathbb{R}^2. \tag{16}$$

We observe immediately that

$$V \in C^{1,1}(\mathbb{R}^2), \tag{17}$$

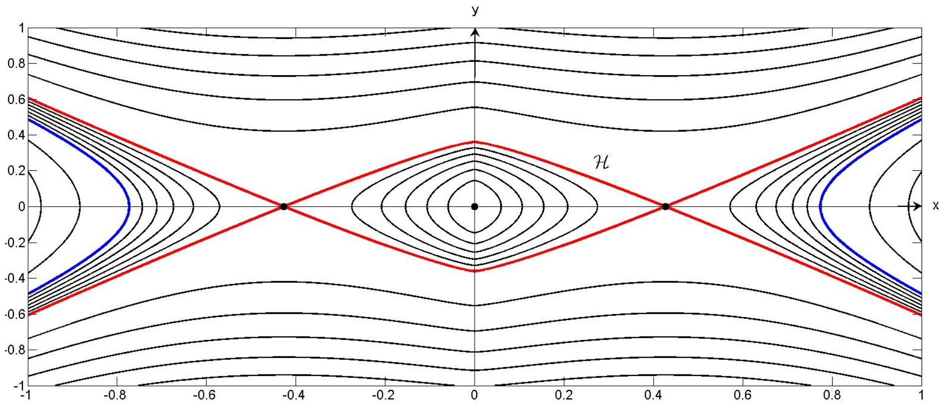


Figure 1: A qualitative sketch of the level curves of $V$

with

$$\nabla V(x,y) = \left( \frac{-1}{(1-p)} x + x|x|^{p-1}, y \right) \quad \forall (x,y) \in \mathbb{R}^2. \tag{18}$$

We now examine the structure of the level curves of $V$ in $\mathbb{R}^2$, namely, the family of curves in $\mathbb{R}^2$ defined by

$$V(x,y) = c, \tag{19}$$

for $-\infty < c < \infty$. It is straightforward to establish that the family of level curves of $V$ are qualitatively as sketched in Figure 1, with $\mathcal{H}$ representing the two level curves connecting $(-(1-p)^{1/1-p}, 0)$ to $((1-p)^{1/1-p}, 0)$ and enclosing the origin. In Figure 3.1, on the red curve $V = (1-p)^{2/(1-p)}/(2(1+p))$, whilst on the blue curves $V = 0$. At $(\pm(1-p)^{1/(1-p)}, 0)$ then $V = (1-p)^{2/(1-p)}/(2(1+p))$, whilst at $(0,0)$ then $V = 0$. Inside $\mathcal{H}$, the level curves are simple closed curves concentric with the origin $(0,0)$, and $V$ is increasing from $V = 0$ at the origin $(0,0)$, as each level curve is crossed, when moving out from the origin $(0,0)$ to the boundary curve $\mathcal{H}$, on which $V = (1-p)^{2/(1-p)}/(2(1+p))$. Thus, inside $\mathcal{H}$, $V$ has a minimum at the origin $(0,0)$ and is increasing on moving radially away from the origin $(0,0)$ to the boundary $\mathcal{H}$. On the level curves exterior and above or below $\mathcal{H}$, then $V > (1-p)^{2/(1-p)}/(2(1+p))$, whilst on the level curves to the left and right side of $\mathcal{H}$, then $V < (1-p)^{2/(1-p)}/(2(1+p))$, with $V = 0$ on the blue level curves. We now focus on the level curves of $V$ on and inside $\mathcal{H}$, which have

$$0 \le c \le c^*(p), \tag{20}$$

where

$$c^*(p) = \frac{(1-p)^{2/(1-p)}}{2(1+p)}. \tag{21}$$

These are concentric closed curves surrounding the origin $(0,0)$. We will label the interior of the level curve $V = c$ by $D_c$, with the level curve $V = c$ labelled as $\partial D_c$, for $0 \le c \le c^*(p)$. In addition, we label the set

$$\bar{D}'_{c^*(p)} = \bar{D}_{c^*(p)} \backslash \{(\pm(1-p)^{1/(1-p)}, 0), (0,0)\}.$$

Now let $(x^*(\eta), y^*(\eta))$ be any solution to (S) for $\eta \in [-E, E]$ (any $E > 0$) with zero-value $(\alpha, \beta) \in \mathbb{R}^2$, and define $F : [-E, E] \to \mathbb{R}$ as,

$$F(\eta) = V(x^*(\eta), y^*(\eta)) \quad \forall \eta \in [-E, E]. \tag{22}$$

Then $F \in C^1([-E, E])$, and via (9), (10) and (14),

$$\begin{aligned} F'(\eta) &= \nabla V(x^*(\eta), y^*(\eta)).(x^{*'}(\eta), y^{*'}(\eta)) \\ &= \nabla V(x^*(\eta), y^*(\eta)).\mathbf{Q}(x^*(\eta), y^*(\eta), \eta) \quad \forall \eta \in [-E, E]. \end{aligned}$$

It then follows, via (18) and (14) that,

$$F'(\eta) = -\frac{1}{2}\eta(y^*(\eta))^2 \quad \forall \eta \in [-E, E]. \tag{23}$$

It follows from (23) that

$$F(\eta) \text{ is non-increasing for } \eta \in [0, E], \tag{24}$$
$$F(\eta) \text{ is non-decreasing for } \eta \in [-E, 0]. \tag{25}$$

We can now establish the following,

**Lemma 2.** *Let $(x^*(\eta), y^*(\eta))$ be any solution to (S) on $[-E, E]$ (any $E > 0$) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Then*

$$(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in [-E, E]\backslash\{0\},$$

*where $c = V(\alpha,\beta)$.*

*Proof.* Let the zero-value $(\alpha,\beta) \in \partial D_c\backslash\{\pm((1-p)^{\frac{1}{1-p}}, 0)\}$ with $0 < c = V(\alpha,\beta) \leq c^*(p)$. We first consider the case when $\beta \neq 0$. It follows from (23)-(25) that,

$$F(\eta) < F(0) \quad \forall \eta \in [-E, E]\backslash\{0\}. \tag{26}$$

Therefore, via (26),

$$V(x^*(\eta), y^*(\eta)) < c \quad \forall \eta \in [-E, E]\backslash\{0\},$$

and so

$$(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in [-E, E]\backslash\{0\},$$

as required. Now consider the case when $\beta = 0$. Then $0 < |\alpha| < (1-p)^{1/(1-p)}$ and therefore, via (10) $y^{*'}(0) \neq 0$ after which a similar argument completes the proof. □

We now have:

**Theorem 3.** *For each $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$, then (S) with zero-value $(\alpha,\beta)$ has a solution $(x^*(\eta), y^*(\eta))$ on $[-E, E]$ (any $E > 0$). Moreover, every such solution satisfies $(x^*(\eta), y^*(\eta)) \in D_c$ for all $\eta \in [-E, E]\backslash\{0\}$, where $c = V(\alpha,\beta)$.*

*Proof.* For any $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$, Lemma 2 establishes that (S) with zero-value $(\alpha,\beta)$ is a priori bounded. The result then follows by a finite number of applications of the Cauchy-Peano Local Existence Theorem (see [24, Chapter 1, Theorem 1.2]), with $\delta = 1/(1+M)$ and

$$M = \max_{(x,y,\eta)\in R'} |\mathbf{Q}(x, y, \eta)|$$

whilst

$$R' = \{(x, y, \eta) \in \mathbb{R}^3 : |x| \leq 2(1-p)^{1/(1-p)},\ |y| \leq 2\sqrt{2c^*(p)},\ |\eta| \leq 2E\}.$$

The final statement follows immediately from Lemma 2. □

We can now establish a global existence result for (S), namely

**Corollary 4.** *For $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$ then (S) with zero-value $(\alpha,\beta)$ has a solution $(x^*(\eta), y^*(\eta))$ on $\mathbb{R}$. Moreover, every such solution satisfies $(x^*(\eta), y^*(\eta)) \in D_c$ for all $\eta \in \mathbb{R}\backslash\{0\}$, where $c = V(\alpha,\beta)$.*

*Proof.* Since Theorem 3 holds for any $E > 0$, the result follows immediately. □

**Remark 2.** Let $(x^*(\eta), y^*(\eta))$ be any solution to (S) on $[-E, E]$ with zero-value $(0,0)$. It follows from (16), (22) and (23) that

$$V(x^*(\eta), y^*(\eta)) = F(\eta) \le F(0) = V(0,0) = 0 \quad \forall \eta \in [-E, E]. \tag{27}$$

Thus $(x^*(\eta), y^*(\eta)) \in \mathcal{S}$ for all $\eta \in [-E, E]$, with $\mathcal{S}$ being a connected subset of

$$\{(x,y) \in \mathbb{R}^2 : V(x,y) \le 0\}$$

for which $(0,0) \in \mathcal{S}$. It follows that $\mathcal{S} = \{(0,0)\}$ and so $(x^*(\eta), y^*(\eta)) = (0,0)$ for all $\eta \in [-E, E]$. We conclude that the unique solution to (S) with zero-value $(0,0)$ is given by,

$$(x^*(\eta), y^*(\eta)) = (0,0) \quad \forall \eta \in \mathbb{R}.$$

We next introduce the function $H : \mathbb{R} \to \mathbb{R}$ such that

$$H(x) = \frac{1}{(1-p)} x - x|x|^{p-1} \quad \forall x \in \mathbb{R}, \tag{28}$$

and observe that

$$H \in C(\mathbb{R}). \tag{29}$$

We have,

**Lemma 5.** *Let $(\alpha, \beta) \in \bar{D}'_{c^*(p)}$, and let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value $(\alpha, \beta)$. Then*

$$y^*(\eta) \to 0 \text{ as } |\eta| \to \infty.$$

*Proof.* We establish the result for $\eta \to \infty$; the result for $\eta \to -\infty$ follows similarly. Now, from (10),

$$y^{*\prime}(\eta) = H(x^*(\eta)) - \frac{1}{2}\eta y^*(\eta) \quad \forall \eta \in [0, \infty). \tag{30}$$

It then follows from (30) that,

$$y^*(\eta) = \beta e^{-\frac{1}{4}\eta^2} + e^{-\frac{1}{4}\eta^2} \int_0^\eta H(x^*(s)) e^{\frac{1}{4}s^2} ds \quad \forall \eta \in [0, \infty). \tag{31}$$

Thus,

$$|y^*(\eta)| \le |\beta| e^{-\frac{1}{4}\eta^2} + e^{-\frac{1}{4}\eta^2} \int_0^\eta |H(x^*(s))| e^{\frac{1}{4}s^2} ds \quad \forall \eta \in [0, \infty). \tag{32}$$

However, via Corollary 4, $(x^*(\eta), y^*(\eta)) \in \bar{D}_{c^*(p)}$ for $\eta \in [0, \infty)$, and so, via (29), there exists a constant $M_H \ge 0$ such that

$$|H(x^*(s))| \le M_H \quad \forall s \in [0, \infty). \tag{33}$$

It then follows from (32) and (33) that

$$|y^*(\eta)| \le |\beta| e^{-\frac{1}{4}\eta^2} + M_H e^{-\frac{1}{4}\eta^2} \int_0^\eta e^{\frac{1}{4}s^2} ds \quad \forall \eta \in [0, \infty). \tag{34}$$

Now a simple application of Watson's Lemma (see [25, Proposition 2.1]), gives,

$$\int_0^\eta e^{\frac{1}{4}s^2} ds \sim \frac{2}{\eta} e^{\frac{1}{4}\eta^2} \text{ as } \eta \to \infty. \tag{35}$$

We then have, via (34) and (35), that

$$|y^*(\eta)| \le |\beta| e^{-\frac{1}{4}\eta^2} + \frac{4M_H}{\eta} \text{ as } \eta \to \infty. \tag{36}$$

It follows from (36) that

$$y^*(\eta) \to 0 \text{ as } \eta \to \infty,$$

as required $\square$

We next have,

**Lemma 6.** *Let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$, and $F : \mathbb{R} \to \mathbb{R}$ as in* (22)*. Then $F(\eta)$ is non-increasing for $\eta \in (0,\infty)$ and non-decreasing for $\eta \in (-\infty, 0)$, with*

$$F(\eta) \to \begin{cases} F_\infty & \text{as } \eta \to \infty \\ F_{-\infty} & \text{as } \eta \to -\infty \end{cases}$$

*where $F_\infty, F_{-\infty} \in [0, F(0))$.*

*Proof.* We observe from Corollary 4 that

$$(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in \mathbb{R}\backslash\{0\}, \tag{37}$$

with $c = V(\alpha,\beta) = F(0)$, and so,

$$0 \leq F(\eta) < F(0) \quad \forall \eta \in \mathbb{R}\backslash\{0\}. \tag{38}$$

In addition, it follows from (38), (24) and (25), since $F \in C^1(\mathbb{R})$, that there exist $F_\infty, F_{-\infty} \in \mathbb{R}$, such that

$$F(\eta) \to \begin{cases} F_\infty & \text{as } \eta \to \infty \\ F_{-\infty} & \text{as } \eta \to -\infty \end{cases}$$

where $F_\infty, F_{-\infty} \in [0, F(0))$, as required. □

We now have,

**Theorem 7.** *Let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Then,*

$$(x^*(\eta), y^*(\eta)) \to (0,0) \text{ as } |\eta| \to \infty.$$

*Proof.* We establish the result for $\eta \to \infty$. The result for $\eta \to -\infty$ follows similarly. We first recall from Corollary 4 that,

$$(x^*(\eta), y^*(\eta)) \in D_{c^*(p)} \quad \forall \eta \in \mathbb{R}\backslash\{0\}, \tag{39}$$

and from Lemma 5 that,

$$y^*(\eta) \to 0 \text{ as } \eta \to \infty. \tag{40}$$

In addition, we have from Lemma 6 that,

$$V(x^*(\eta), y^*(\eta)) \to F_\infty \text{ as } \eta \to \infty \tag{41}$$

for some $F_\infty \in [0, c^*(p))$. It follows from (39)-(41) that

$$x^*(\eta) \to x_\infty \text{ or } x^*(\eta) \to -x_\infty \text{ as } \eta \to \infty \tag{42}$$

where $x_\infty$ is the single non-negative root of

$$V(x,0) = F_\infty \text{ with } x \in [0, (1-p)^{1/(1-p)}).$$

Without loss of generality we will suppose that

$$(x^*(\eta), y^*(\eta)) \to (x_\infty, 0) \text{ as } \eta \to \infty. \tag{43}$$

However, it follows from (10) that,

$$y^*(\eta) = \beta e^{-\frac{1}{4}\eta^2} + e^{-\frac{1}{4}\eta^2} \int_0^\eta H(x^*(s)) e^{\frac{1}{4}s^2} ds \quad \eta \in [0,\infty) \tag{44}$$

with $H : \mathbb{R} \to \mathbb{R}$ given by (28), and

$$H(x_\infty) \leq 0. \tag{45}$$

Using (42), it is straightforward to establish that, when,

$$H(x_\infty) < 0, \tag{46}$$

then from (44),

$$y^*(\eta) \sim \frac{2H(x_\infty)}{\eta} \text{ as } \eta \to \infty. \tag{47}$$

In addition, from (9), we have,

$$x^*(\eta) = \alpha + \int_0^\eta y^*(s)ds \quad \forall \eta \in [0, \infty), \tag{48}$$

which gives, via (47), that

$$x^*(\eta) \sim 2H(x_\infty) \log \eta, \quad \text{ as } \eta \to \infty,$$

which contradicts (42). We conclude that (46) cannot hold, and so, via (45), we must have

$$H(x_\infty) = 0, \tag{49}$$

which, since $x_\infty \in [0, (1-p)^{1/(1-p)})$, requires $x_\infty = 0$. It then follows from (43) that,

$$(x^*(\eta), y^*(\eta)) \to (0, 0) \text{ as } \eta \to \infty,$$

as required. □

We conclude from Corollary 4 and Theorem 7 that the problem (S) has a two parameter family of nontrivial, distinct homoclinic connections on the equilibrium point $(0, 0)$, parametrized by $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$ which we will denote by $w_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ for each $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Here $w = w_{\alpha,\beta}(\eta)$, $\eta \in \mathbb{R}$, has zero-values $w(0) = \alpha$, $w'(0) = \beta$. Moreover,

$$(w_{\alpha,\beta}(\eta), w'_{\alpha,\beta}(\eta)) \in D_{V(\alpha,\beta)} \quad \forall \eta \in \mathbb{R}\backslash\{0\}.$$

Additionally, note that $w_{0,\beta}(\eta)$ is an odd function of $\eta$ whilst $w_{\alpha,0}(\eta)$ is an even function of $\eta$. Furthermore, it also follows from the comments below (13) that $w_{\alpha,\beta}(\eta)$ must be two signed for $\eta \in \mathbb{R}$.

### *3.2. Decay Bounds and Estimates*

In this section, we establish results concerning the rate of decay to zero of $w_{\alpha,\beta}(\eta)$ as $\eta \to \pm\infty$. Specifically, we establish algebraic bounds on the rate of decay of $w_{\alpha,\beta}(\eta)$ as $\eta \to \pm\infty$, and hence, determine that $w_{\alpha,\beta} \in L_q(\mathbb{R})$ for each $q > (1-p)/2$. From these bounds we may infer that the corresponding solution to [CP], say $u_{\alpha,\beta} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, satisfies $u(\cdot, t) \in L_q(\mathbb{R})$ for each $t \in [0, \infty)$ and $q > (1-p)/2$. To complement the algebraic bounds, we also provide a rational asymptotic approximation to the decay rate of $w_{\alpha,\beta}(\eta)$ as $\eta \to \pm\infty$, which, in fact suggests exponential decay as $\eta \to \pm\infty$.

To begin, observe that $w = w_{\alpha,\beta}(\eta)$ for $\eta \in \mathbb{R}$, via (6), satisfies

$$(e^{\frac{1}{4}\eta^2} w')' = H(w)e^{\frac{1}{4}\eta^2} \quad \forall \eta \in \mathbb{R}.$$

It follows from two successive integrations, that

$$w'(\eta) = \beta e^{-\frac{1}{4}\eta^2} + e^{-\frac{1}{4}\eta^2} \int_0^\eta H(w(s))e^{\frac{1}{4}s^2} ds \quad \forall \eta \in \mathbb{R} \tag{50}$$

whilst,

$$w(\eta) = \alpha + \int_0^\eta \beta e^{-\frac{1}{4}t^2} dt + \int_0^\eta e^{-\frac{1}{4}t^2} \int_0^t H(w(s))e^{\frac{1}{4}s^2} ds dt \quad \forall \eta \in \mathbb{R}. \tag{51}$$

We now have,

**Proposition 8.** *Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Suppose that*

$$|w(\eta)| \leq \frac{c_1}{(1+|\eta|)^{\sigma}} \quad \forall \eta \in \mathbb{R}.$$

*with $\sigma \geq 0$ and $c_1 > 0$ (independent of $\alpha$ and $\beta$). Then, there exists $c_2 > 0$, which depends on $c_1$, $\sigma$ and $p$, (independent of $\alpha$ and $\beta$) such that,*

$$|w'(\eta)| \leq \frac{c_2}{(1+|\eta|)^{\sigma p+1}} \quad \forall \eta \in \mathbb{R}.$$

*Proof.* We give a proof for $\eta \geq 0$; the result for $\eta < 0$ follows similarly. Observe that

$$|H(w(\eta))| = \left| \frac{1}{(1-p)} w(\eta) - |w(\eta)|^{p-1} w(\eta) \right| \leq \frac{c_1^p}{(1+\eta)^{\sigma p}} \quad \forall \eta \in [0,\infty), \tag{52}$$

since, via Corollary 4, $|w(\eta)| < (1-p)^{\frac{1}{(1-p)}}$ for $\eta \in [0,\infty)$. Thus, via (50) and (52), we have,

$$|w'(\eta)| \leq |\beta| e^{-\frac{1}{4}\eta^2} + c_1^p e^{-\frac{1}{4}\eta^2} \int_0^{\eta} \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \quad \forall \eta \in [0,\infty). \tag{53}$$

Now, the second term on the right hand side of (53) is a non-negative continuous function for $\eta \in [0,\infty)$, with asymptotic form,

$$c_1^p e^{-\frac{1}{4}\eta^2} \int_0^{\eta} \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \sim \frac{2c_1^p}{\eta^{\sigma p+1}} \text{ as } \eta \to \infty.$$

It follows that,

$$c_1^p e^{-\frac{1}{4}\eta^2} \int_0^{\eta} \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \leq \frac{4c_1^p}{\eta^{\sigma p+1}} \text{ as } \eta \to \infty.$$

We conclude that there exists a positive constant $c_2$, depending upon $c_1$, $p$, and $\sigma$, such that

$$|w'(\eta)| \leq \frac{c_2}{(1+\eta)^{\sigma p+1}} \quad \forall \eta \in [0,\infty),$$

as required. □

We next have,

**Proposition 9.** *Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Then,*

$$(w(\eta), w'(\eta)) \to (0,0) \text{ as } \eta \to \pm\infty,$$

*and moreover,*

$$|w'(\eta)| \leq \frac{c_2}{(1+|\eta|)} \quad \forall \eta \in \mathbb{R},$$

*with $c_2 > 0$ dependent upon $p$ (independent of $\alpha$ and $\beta$).*

*Proof.* The first conclusion follows directly from Theorem 7. Additionally, it follows from Corollary 4 that

$$(w(\eta), w'(\eta)) \in \mathcal{H} \quad \forall \eta \in \mathbb{R},$$

and hence, it follows from Proposition 8 (with $\sigma = 0$, $c_1 = (1-p)^{1/(1-p)}$) that

$$|w'(\eta)| \leq \frac{c_2}{(1+|\eta|)} \quad \forall \eta \in \mathbb{R},$$

as required. □

We now demonstrate that every solution $w : \mathbb{R} \to \mathbb{R}$ to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$ decays to zero as $\eta \to \pm\infty$, with decay rate which is at least algebraic in $\eta$ as $\eta \to \pm\infty$. In particular, we demonstrate that $w : \mathbb{R} \to \mathbb{R}$ is contained in $L^q(\mathbb{R})$ for any $q > (1-p)/2$. The proof is based on the decay bounds obtained in [8].

**Theorem 10.** *Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Then, for any $\epsilon > 0$, there exists $c_{1\epsilon}, c_{2\epsilon} > 0$ (dependent generally on $\alpha$, $\beta$, $p$ and $\epsilon$) such that*

$$|w(\eta)| < \frac{c_{1\epsilon}}{(1+|\eta|)^{\frac{2}{(1-p)}-\epsilon}} \quad \forall \eta \in \mathbb{R},$$

$$|w'(\eta)| < \frac{c_{2\epsilon}}{(1+|\eta|)^{\frac{(1+p)}{(1-p)}-\epsilon}} \quad \forall \eta \in \mathbb{R}.$$

*Proof.* We give a proof for $\eta \geq 0$; the argument for $\eta < 0$ follows similarly. Observe on multiplying (6) by $\eta^{-1}w(\eta)$, we have,

$$\frac{1}{\eta}\left[|w(\eta)|^{1+p} - \frac{(w(\eta))^2}{(1-p)}\right] = -\left[\frac{(w(\eta))^2}{4} + \frac{w(\eta)w'(\eta)}{\eta}\right]' + \frac{(w'(\eta))^2}{\eta} - \frac{w(\eta)w'(\eta)}{\eta^2} \tag{54}$$

for $\eta \in (0,\infty)$. Additionally, via Proposition 9, it follows that there exists $\eta_* \in (0,\infty)$ such that,

$$|w(\eta)| \leq \left(\frac{2p(1-p)}{(1+p)}\right)^{\frac{1}{(1-p)}} \quad \forall \eta \in [\eta_*, \infty), \tag{55}$$

and for $F : [0,\infty) \to \mathbb{R}$ given by

$$F(\eta) = V(w(\eta), w'(\eta)) \quad \forall \eta \in [0,\infty),$$

that

$$0 \leq F(\eta) \leq \left(\frac{4(c(p))^{\frac{2}{(1+p)}}}{C(p)}\right)^{(1+p)/(1-p)} \quad \eta \in [\eta_*, \infty), \tag{56}$$

where

$$c(p) = \frac{1}{(1+p)} - \frac{1}{2}, \text{ and } C(p) = \frac{2(1+p)}{(1-p)} + 1. \tag{57}$$

Thus, it follows from (54) that

$$\begin{aligned}
\frac{F(\eta)}{\eta} &= \frac{(w'(\eta))^2}{2\eta} + \frac{1}{\eta}\left[-\frac{(w(\eta))^2}{2(1-p)} + \frac{|w(\eta)|^{1+p}}{(1+p)}\right] \\
&\leq \frac{(w'(\eta))^2}{2\eta} + \frac{1}{\eta}\left[-\frac{(w(\eta))^2}{(1-p)} + |w(\eta)|^{1+p}\right] \\
&= \frac{3(w'(\eta))^2}{2\eta} - \left[\frac{(w(\eta))^2}{4} + \frac{w(\eta)w'(\eta)}{\eta}\right]' - \frac{w(\eta)w'(\eta)}{\eta^2},
\end{aligned} \tag{58}$$

for $\eta \in [\eta_*, \infty)$. Since $F(\eta) \geq 0$ for all $\eta \in [\eta^*, \infty)$, together with the decay estimates in Proposition 9, it follows that we may integrate inequality (58) from $\eta$ $(\geq \eta^*)$ to $l$, and then allow $l \to \infty$, to obtain,

$$\int_\eta^\infty \frac{F(t)}{t}dt \leq \frac{(w(\eta))^2}{4} + \frac{2}{\eta}\sup_{t\geq\eta}|w(t)w'(t)| + \frac{3}{2}\int_\eta^\infty \frac{(w'(t))^2}{t}dt \tag{59}$$

for $\eta \in [\eta_*, \infty)$. We also note, that since, via Corollary 4, $|w(\eta)| < (1-p)^{1/(1-p)}$, we have,

$$F(\eta) \geq |w(\eta)|^{1+p}c(p) \geq 0, \tag{60}$$

for $\eta \in [\eta_*, \infty)$. It therefore follows from (59) and (60) that

$$0 \le \int_\eta^\infty \frac{F(t)}{t} dt \le \frac{1}{4} \left( \frac{F(\eta)}{c(p)} \right)^{\frac{2}{(1+p)}} + \frac{2}{\eta} \sup_{t \ge \eta} |w(t) w'(t)| + \frac{3}{2} \int_\eta^\infty \frac{(w'(t))^2}{t} dt \tag{61}$$

for $\eta \in [\eta_*, \infty)$. We observe that the right hand side of (61) is uniformly bounded for $\eta \in [\eta_*, \infty)$ via Proposition 9.

Now suppose that there exists $k > 0$ such that

$$F(\eta) \le \frac{k}{\eta^\sigma} \quad \forall \eta \in [\eta_*, \infty) \tag{62}$$

for some $\sigma \ge 0$ (note that (62) holds when $\sigma = 0$ via Proposition 9). Then, via (60), it follows that there exists $c_1 > 0$ such that

$$|w(\eta)| \le \frac{c_1}{\eta^{\frac{\sigma}{(1+p)}}} \quad \forall \eta \in [\eta_*, \infty) \tag{63}$$

and so, via Proposition 8, there exists $c_2 > 0$ such that

$$|w'(\eta)| \le \frac{c_2}{\eta^{\frac{\sigma p}{(1+p)}+1}} \quad \forall \eta \in [\eta_*, \infty). \tag{64}$$

Thus, it follows from (61)-(64) and (56), that there exists $c_3, c_4, c_5 > 0$ such that

$$\begin{aligned} \int_\eta^\infty \frac{F(t)}{t} dt &\le \frac{1}{4} \left( \frac{F(\eta)}{c(p)} \right)^{\frac{2}{(1+p)}} + \frac{c_3}{\eta^{\sigma+2}} + \frac{c_4}{\eta^{\frac{2\sigma p}{(1+p)}+2}} \\ &\le \frac{F(\eta)}{C(p)} + \frac{c_5}{\eta^{\frac{2\sigma p}{(1+p)}+2}} \end{aligned} \tag{65}$$

for $\eta \in [\eta_*, \infty)$. Upon setting $G : [\eta_*, \infty) \to \mathbb{R}$ to be

$$G(\eta) = \int_\eta^\infty \frac{F(t)}{t} dt \quad \forall \eta \in [\eta_*, \infty),$$

it follows from (65) that $G$ satisfies,

$$\left( t^{C(p)} G(t) \right)' \le \frac{c_6}{t^{\frac{2\sigma p}{(1+p)}+3-C(p)}} \quad \forall t \in [\eta_*, \infty), \tag{66}$$

with $c_6 > 0$ constant. An integration of (66) gives

$$G(\eta) \le \frac{c_7}{\eta^{\frac{2\sigma p}{(1+p)}+2}} + \frac{c_8}{\eta^{C(p)}} \quad \forall \eta \in [\eta_*, \infty), \tag{67}$$

with $c_7, c_8 > 0$ constants. Also, recalling, via Lemma 6, that $F(\eta)$ is non-increasing on $[\eta^*, \infty)$, we have,

$$G(\eta) \ge \int_\eta^{2\eta} \frac{F(t)}{t} dt \ge \frac{1}{2} F(2\eta), \quad \forall \eta \in [\eta_*, \infty). \tag{68}$$

Thus, it follows from (67) and (68) that there exist constants $c_9, c_{10} > 0$ such that

$$F(\eta) \le \frac{c_9}{\eta^{\frac{2\sigma p}{(1+p)}+2}} + \frac{c_{10}}{\eta^{C(p)}} \quad \forall \eta \in [\eta_*, \infty). \tag{69}$$

Since (62) holds for $\sigma = 0$, it follows that there exists sequences $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{k_n\}_{n \in \mathbb{N}}$ given by

$$\sigma_1 = 0, \quad \sigma_{n+1} = \min \left\{ \frac{2\sigma_n p}{(1+p)} + 2, \ C(p) \right\} \tag{70}$$

such that

$$F(\eta) \leq \frac{k_n}{\eta^{\sigma_n}} \quad \forall \eta \in [\eta_*, \infty). \tag{71}$$

We obtain from (70) and (57) that,

$$\sigma_n = \frac{2(1+p)}{(1-p)} - \frac{4p}{(1-p)}\left(\frac{2p}{(1+p)}\right)^{n-2} \quad \forall n \in \mathbb{N}$$

and hence $\sigma_n$ is increasing with

$$\sigma_n \to \frac{2(1+p)}{(1-p)} \quad \text{as } n \to \infty. \tag{72}$$

Therefore it follows, via (60) and (70)-(72), that for each $\epsilon > 0$ there exists $c_{1\epsilon} > 0$ such that

$$|w(\eta)| \leq \frac{c_{1\epsilon}}{(1+\eta)^{\frac{2}{(1-p)}-\epsilon}} \quad \forall \eta \in [0, \infty), \tag{73}$$

recalling that $w(\eta)$ is bounded on $[0, \eta^*]$. The bound on $|w'(\eta)|$ follows immediately from (73) and Proposition 8. □

The algebraic bounds in Theorem 10 are the tightest decay rates we have been able to establish rigorously. However, the following asymptotic argument indicates that, in fact, $w = w_{\alpha,\beta}(\eta)$ decays exponentially in $\eta$ as $|\eta| \to \infty$, accompanied by rapid oscillatory behaviour. To this end, we now consider the asymptotic structure of $w = w_{\alpha,\beta}(\eta)$ as $\eta \to \infty$, with the same structure following as $\eta \to -\infty$. Now, for $\eta >> 1$, then $w = w_{\alpha,\beta}(\eta)$ satisfies,

$$w'' + \frac{1}{2}\eta w' + w|w|^{p-1} - \frac{1}{(1-p)}w = 0 \quad \eta >> 1 \tag{74}$$

$$w(\eta),\ w'(\eta) \to 0 \quad \text{as } \eta \to \infty, \tag{75}$$

via (6) and Proposition 9. On using (75), the dominant form of (74) when $\eta >> 1$ is

$$w'' + w|w|^{p-1} = 0. \tag{76}$$

Every solution to (76) is periodic and may be written (up to translation in $\eta$) as,

$$w(\eta, a) = aW\left(a^{-\frac{1}{2}(1-p)}\eta\right), \quad \forall \eta \in \mathbb{R}, \tag{77}$$

where $a \in \mathbb{R}^+$ is a parameter and $W : \mathbb{R} \to \mathbb{R}$ is that unique periodic function which satisfies the problem,

$$W'' + W|W|^{p-1} = 0, \quad \zeta \in \mathbb{R} \tag{78}$$

$$W(0) = 1, \quad W'(0) = 0. \tag{79}$$

The period of $W(\zeta)$ is given by

$$T(p) = 2^{3/2}(1+p)^{1/2}\int_0^1 \frac{d\lambda}{(1-\lambda^{(1+p)})^{1/2}} \tag{80}$$

whilst,

$$W(\zeta) = -W\left(\frac{1}{2}T(p) - \zeta\right) = W(-\zeta) \quad \forall \zeta \in \mathbb{R}. \tag{81}$$

Via an integration, the solution to (78)-(79) satisfies

$$\frac{(W'(\eta))^2}{2} + \frac{|W(\eta)|^{1+p}}{(1+p)} = \frac{1}{(1+p)} \quad \forall \eta \in \mathbb{R},$$

which represents a periodic orbit in the $(W, W')$ phase plane, as illustrated in Figure 3.2. It follows from (77) that $w(\eta, a)$ has amplitude $a > 0$ and period

$$T_a(p) = a^{\frac{1}{2}(1-p)}T(p). \tag{82}$$

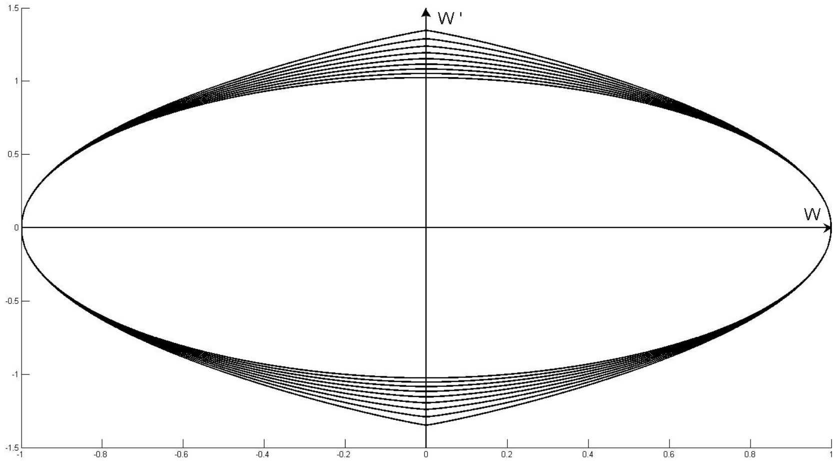


Figure 2: Phase paths for solutions to (78)-(79) for $p_k = (0.1)k$ for $k = 1...9$. Here the phase path for $p_k$ encloses the phase path for $p_{k+1}$ for $k = 1...8$.

For any fixed $a \in \mathbb{R}^+$, (77) cannot represent the asymptotic structure to (74) and (75) since $W$ is periodic. The remaining terms in (74) must induce decay as $\eta \to \infty$. However, we observe from (82) that the oscillations in $w(\eta, a)$ becomes increasingly rapid as the amplitude $a \to 0^+$. This suggests that we seek the asymptotic structure of (74)-(75) as $\eta \to \infty$ in the form,

$$w(\eta) \sim a(\eta) W(a(\eta)^{-\frac{1}{2}(1-p)}\eta) \text{ as } \eta \to \infty, \tag{83}$$

with $a(\eta) > 0$ and,

$$a(\eta), a'(\eta) \to 0 \text{ as } \eta \to \infty. \tag{84}$$

Now, the rate of change of amplitude of oscillation in (83), $a'(\eta)$, approaches zero as $\eta \to \infty$, whilst the frequency of oscillation becomes unbounded as $\eta \to \infty$. We can thus use an averaging approach to determine an evolution equation for the amplitude $a(\eta)$ as $\eta \to \infty$. We substitute (83) into (6) and make use of (78). We then integrate the resulting ordinary differential equation over *one* period of $W(\cdot)$, over which, we may hold $a$ fixed. We obtain the leading order amplitude equation as,

$$a'' + \frac{1}{2}\eta a' - \frac{1}{(1-p)}a = 0, \quad \eta >> 1, \tag{85}$$

$$a(\eta), a'(\eta) \to 0 \text{ as } \eta \to \infty. \tag{86}$$

The linear ordinary differential equation (85) has two basis functions $a_+ : \mathbb{R} \to \mathbb{R}$ and $a_- : \mathbb{R} \to \mathbb{R}$ which have

$$a_+(\eta) \sim \eta^{-\left(1+\frac{2}{(1-p)}\right)} e^{-\frac{1}{4}\eta^2}, \quad a_-(\eta) \sim \eta^{\frac{2}{(1-p)}} \quad \text{as } \eta \to \infty.$$

It follows that

$$a(\eta) \sim A_\infty \eta^{-\left(1+\frac{2}{(1-p)}\right)} e^{-\frac{1}{4}\eta^2} \quad \text{as } \eta \to \infty, \tag{87}$$

with $A_\infty$ being a positive globally determined constant dependent, in general, on $\alpha, \beta$ and $p$. Thus, from (83), we have

$$w_{\alpha,\beta}(\eta) \sim a(\eta) W(a(\eta)^{-\frac{1}{2}(1-p)}\eta) \text{ as } \eta \to \infty, \tag{88}$$

with, $\alpha(\eta)$ having the asymptotic form (87) as $\eta \to \infty$. The same argument leads to the same (up to the constant $A_\infty$) asymptotic structure as $\eta \to -\infty$. As a consequence of (87) and (88), we anticipate that $w_{\alpha,\beta}(\eta)$ decays to zero at a Gaussian rate as $|\eta| \to \infty$, whilst oscillating about zero with a local frequency which increases at a Gaussian rate as $|\eta| \to \infty$. This indicates that, in fact, $w_{\alpha,\beta} \in L^q(\mathbb{R})$ for any $q > 0$.

*3.3. Localized Solutions to [CP]*

Following Corollary 4 and Theorem 7, for each $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$, we have constructed a non-trivial, localized, global solution $u_{\alpha,\beta} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ to [CP], namely,

$$u_{\alpha,\beta}(x,t) = \begin{cases} t^{\frac{1}{(1-p)}} w_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right) & ,(x,t) \in \mathbb{R} \times (0,\infty) \\ 0 & ,(x,t) \in \mathbb{R} \times \{0\}. \end{cases} \tag{89}$$

With this two parameter family of solutions to [CP], each solution is distinct, and is not a spatial translate of any other solution in the family. However, we observe that $u_{\alpha,\beta}(x-x_0,t)$ is also a global solution to [CP] for any fixed $x_0 \in \mathbb{R}$. A trivial calculation from (89) establishes that

$$(u_{\alpha,\beta})_x(x,t) = t^{\frac{1}{(1-p)}-\frac{1}{2}} w'_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right), \tag{90}$$

$$(u_{\alpha,\beta})_t(x,t) = \frac{1}{(1-p)} t^{\frac{1}{(1-p)}-1} \left( w_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right) - \frac{1}{2}(1-p)\left(\frac{x}{t^{1/2}}\right) w'_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right)\right), \tag{91}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, whilst from (1),

$$(u_{\alpha,\beta})_{xx}(x,t) = (u_{\alpha,\beta})_t(x,t) - (u_{\alpha,\beta}|u_{\alpha,\beta}|^{p-1})(x,t), \tag{92}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. It then follows immediately from Theorem 7 that,

$$(u_{\alpha,\beta})_x,\ (u_{\alpha,\beta})_t,\ (u_{\alpha,\beta})_{xx} \to 0 \text{ as } t \to 0^+ \text{ uniformly for } x \in \mathbb{R},$$

and so, in fact,

$$u_{\alpha,\beta} \in L^\infty(\mathbb{R} \times [0,T]) \cap C(\mathbb{R} \times [0,T]) \cap C^{2,1}(\mathbb{R} \times [0,T]). \tag{93}$$

It follows from (93) that for each $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$, and any $\tau > 0$, then $u^\tau_{\alpha,\beta} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that

$$u^\tau_{\alpha,\beta}(x,t) = \begin{cases} (t-\tau)^{\frac{1}{(1-p)}} w_{\alpha,\beta}\left(\frac{x}{(t-\tau)^{1/2}}\right) & ,(x,t) \in \mathbb{R} \times (\tau,\infty) \\ 0 & ,(x,t) \in \mathbb{R} \times [0,\tau] \end{cases}$$

is also a non-trivial, localized, global solution to [CP]. Finally, we observe, via Theorem 10 that for each $q > (1-p)/2$, then $u_{\alpha,\beta}(\cdot,t) \in L^q(\mathbb{R}$ for each $t \geq 0$. Moreover, (87) and (88) suggest that the localization is Gaussian in $x$ for each $t > 0$.

## 4. Heteroclinic Connections

In this section we establish the existence of at least one heteroclinic connection for (S) from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$.

*4.1. Existence*

We first consider solutions to the problem (S) for $\eta \in [0,\infty)$ and which remain in the region $\Omega \subset \mathbb{R}^2$, given as

$$\Omega = \left\{(x,y) : 0 < x < (1-p)^{1/(1-p)}, y > 0\right\} \tag{94}$$

with boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$. We also define the following subset of $\partial\Omega$, namely,

$$\partial\Omega_1 = \{(x,y) : x = 0, y > 0\}. \tag{95}$$

Specifically, we consider (S) for $\eta \in [0,\infty)$ and demonstrate that there exists a solution $(x,y) : [0,\infty) \to \overline{\Omega}$ with zero-value $(0,\beta) \in \partial\Omega_1$ and which satisfies

$$(x(\eta), y(\eta)) \in \Omega \quad \forall \eta \in (0,\infty), \tag{96}$$

$$(x(\eta), y(\eta)) \to ((1-p)^{1/(1-p)}, 0) \quad \text{as } \eta \to \infty. \tag{97}$$

To begin with, it is readily established that for each zero-value $(0,\beta) \in \partial\Omega_1$, then (S) has a local solution $(x,y) : [0,\delta] \to \mathbb{R}^2$ (for some $\delta > 0$). Moreover, $(x(\eta), y(\eta)) \in \Omega$ for $\eta \in (0,\delta]$, and $x(\eta)$ is monotone increasing whilst $y(\eta)$ is monotone decreasing, with $\eta \in (0,\delta]$. It is then straightforward to establish that $(x(\eta), y(\eta))$ can be *uniquely* continued beyond $\eta = \delta$ and must satisfy one of the following three possibilities:

(i) There exists $\eta_\beta > 0$ such that $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \eta_\beta)$ and $(x(\eta_\beta), y(\eta_\beta)) = ((1-p)^{\frac{1}{1-p}}, y_\beta)$ with $0 < y_\beta < \beta$, whilst $x'(\eta_\beta) = y_\beta > 0$, and so there exists $\epsilon_\beta > 0$ such that $(x(\eta), y(\eta)) \notin \bar{\Omega} \cup (\{0\} \times \mathbb{R})$ for $\eta \in (\eta_\beta, \eta_\beta + \epsilon_\beta]$.

(ii) There exists $\eta_\beta > 0$ such that $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \eta_\beta)$ and $(x(\eta_\beta), y(\eta_\beta)) = (x_\beta, 0)$ with $0 < x_\beta < (1-p)^{\frac{1}{1-p}}$, whilst $y'(\eta_\beta) < 0$ and so there exists $\epsilon_\beta > 0$ such that $(x(\eta), y(\eta)) \notin \bar{\Omega} \cup (\{0\} \times \mathbb{R})$ for $\eta \in (\eta_\beta, \eta_\beta + \epsilon_\beta]$.

(iii) $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \infty)$ and $(x(\eta), y(\eta)) \to ((1-p)^{\frac{1}{1-p}}, 0)$ as $\eta \to \infty$.

Our aim now is to obtain a uniqueness result for (S) with zero-value in $\partial\Omega_1$, and from this a continuous dependence result. This is non-trivial, since $\mathbf{Q}$ in (14) is not locally Lipschitz continuous in any neighborhood of $(0,\beta) \in \partial\Omega_1$, and so standard uniqueness and continuous dependence theory fail to apply. To begin with, we provide a local a priori bound for any solution of (S) with zero-value $(0,\beta) \in \partial\Omega_1$.

**Proposition 11.** *Let $(x,y) : [0, \eta_\beta] \to \mathbb{R}^2$ be any solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$ and which satisfies either case (i) or (ii). Then,*

$$\eta_\beta > \min\left\{ \left(\frac{2}{\beta}\right)\left(m_H + \sqrt{m_H^2 + \frac{(\beta)^2}{2}}\right), \ \frac{(1-p)^{1/(1-p)}}{\beta} \right\} = \eta^* \tag{98}$$

*with*

$$m_H = \inf_{\lambda \in [0,(1-p)^{1/(1-p)}]} H(\lambda), \tag{99}$$

*Proof.* Let $(x,y) : [0, \eta_\beta] \to \mathbb{R}^2$ be any solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$, and which satisfies either case (i) or case (ii). Suppose that $\eta_\beta \leq \eta^*$. Since $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \eta_\beta)$, it follows from (10) that

$$\beta + m_H \eta - \frac{\beta}{4}\eta^2 < y(\eta) < \beta \quad \forall \eta \in (0, \eta_\beta]. \tag{100}$$

However, $\eta_\beta \leq \eta^*$ and so, via (100),

$$\frac{\beta}{2} < y(\eta) < \beta \quad \forall \eta \in (0, \eta_\beta]. \tag{101}$$

An integration of (9) using (101), then gives,

$$\frac{\beta\eta}{2} < x(\eta) < \beta\eta \quad \forall \eta \in (0, \eta_\beta]. \tag{102}$$

It finally follows from (101) and (102), since $\eta_\beta \leq \eta^*$, that,

$$x(\eta_\beta) < \beta\eta_\beta \leq \beta\eta^* \leq (1-p)^{\frac{1}{1-p}}, \quad y(\eta_\beta) > \frac{\beta}{2},$$

and so $(x(\eta_\beta), y(\eta_\beta)) \in \Omega$, which is a contradiction. We conclude that $\eta_\beta > \eta^*$, as required. □

Therefore, we have,

**Corollary 12.** *Let $(x,y) : [0, \eta^*] \to \mathbb{R}^2$ be a solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$ with $\eta^*$ given by* (98). *Then,*

$$\frac{\beta\eta}{2} < x(\eta) < (1-p)^{1/(1-p)}, \quad \frac{\beta}{2} < y(\eta) < \beta \quad \forall \eta \in [0, \eta^*],$$

*Proof.* For cases (i) and (ii), the result follows from Proposition 11, with case (iii) following immediately. □

The a priori bounds in Corollary 12, allow us to establish the following local uniqueness result for (S) with zero-value $(0,\beta) \in \partial\Omega_1$. The proof is based on the uniqueness argument in [2].

**Proposition 13.** *The problem (S) with zero-value $(0,\beta) \in \partial\Omega_1$ has at most one solution on $[0,\eta^*]$, with $\eta^* > 0$ given by* (98).

*Proof.* To begin, fix $(0,\beta) \in \partial\Omega_1$. Suppose that $(x,y),(x^*,y^*) : [0,\eta^*] \to \mathbb{R}^2$ are solutions to (S) with zero-value $(0,\beta)$. It follows from Corollary 12 that

$$(x(\eta),y(\eta)),(x^*(\eta),y^*(\eta)) \in \bar{\Omega} \quad \forall \eta \in [0,\eta^*], \tag{103}$$

whilst from Corollary 12,

$$|x(\eta) - x^*(\eta)| < (1-p)^{1/(1-p)}, \quad |y(\eta) - y^*(\eta)| < \beta \quad \forall \eta \in [0,\eta^*]. \tag{104}$$

Additionally, we observe that for $(X,Y) \in [0,(1-p)^{1/(1-p)}] \times [0,\beta]$, then

$$X + X^p + Y < \left(2 + \beta^{1-p}\right)(X+Y)^p, \tag{105}$$

since $0 < p < 1$. Now, via (9) and (10) respectively, we have,

$$|x(\eta) - x^*(\eta)| \le \int_0^\eta |y(s) - y^*(s)| ds \tag{106}$$

$$|y(\eta) - y^*(\eta)| \le \int_0^\eta \left( \frac{1}{(1-p)}|x(s) - x^*(s)| + |x(s) - x^*(s)|^p + \frac{s}{2}|y(s) - y^*(s)| \right) ds \tag{107}$$

for all $\eta \in [0,\eta^*]$. We next introduce $v : [0,\eta^*] \to \mathbb{R}$ as,

$$v(\eta) = |x(\eta) - x^*(\eta)| + |y(\eta) - y^*(\eta)| \quad \forall \eta \in [0,\eta^*]. \tag{108}$$

Therefore, via (103)-(108), it follows that

$$\begin{aligned} v(\eta) &\le \int_0^\eta \left( \frac{1}{(1-p)}|x(s) - x^*(s)| + |x(s) - x^*(s)|^p + \left(\frac{s}{2} + 1\right)|y(s) - y^*(s)| \right) ds \\ &\le \int_0^\eta \frac{1}{(1-p)} \left(\frac{\eta^*}{2} + 1\right) \left(|x(s) - x^*(s)| + |x(s) - x^*(s)|^p + |y(s) - y^*(s)|\right) ds \\ &\le \int_0^\eta \frac{1}{(1-p)} \left(\frac{\eta^*}{2} + 1\right) (2 + \beta^{1-p}) (v(s))^p \, ds \end{aligned} \tag{109}$$

for all $\eta \in [0,\eta^*]$, where the final inequality is due to (104) and (105). Also, via Corollary 12 and (98), $\eta^*$ is dependent on $p$ and $\beta$ only, and hence, it follows from (109) that

$$v(\eta) \le \int_0^\eta K(p,\beta)(v(s))^p \, ds \tag{110}$$

for all $\eta \in [0,\eta^*]$, where the constant $K(p,\beta)$ is given by,

$$K(p,\beta) = \frac{1}{(1-p)} \left(\frac{\eta^*}{2} + 1\right) (2 + \beta^{1-p}).$$

We now introduce the function $\bar{H} : [0,\eta^*] \to \overline{\mathbb{R}}_+$ given by,

$$\bar{H}(\eta) = \int_0^\eta K(p,\beta)(v(s))^p \, ds \quad \forall \eta \in [0,\eta^*]. \tag{111}$$

It follows from (111) that $\bar{H}$ is non-negative, non-decreasing and differentiable on $[0,\eta^*]$, and via (110), satisfies

$$(\bar{H}(s))' \le K(p,\beta)(\bar{H}(s))^p \quad \forall s \in [0,\eta^*]. \tag{112}$$

Upon integrating (112) from 0 to $\eta$, we obtain

$$\bar{H}(\eta) \leq ((1-p)K(p,\beta)\eta)^{1/(1-p)} \quad \forall \eta \in [0,\eta^*] \tag{113}$$

and it follows from (113), (111) and (110) that

$$v(\eta) \leq \delta \quad \forall \eta \in [0,\eta_\delta], \tag{114}$$

where $\delta > 0$ is chosen sufficiently small so that

$$\eta_\delta = \frac{\delta^{1-p}}{(1-p)K(p,\beta)} < \eta^*.$$

Now, from Corollary 12, we have

$$\min\{x^*(\eta), x(\eta)\} \geq \frac{\beta\eta}{2} \quad \forall \eta \in [0,\eta^*]. \tag{115}$$

Moreover, it follows from (14), (115) and the mean value theorem, that there exists $\theta(s) \geq \min\{x^*(s), x(s)\}$, for which,

$$\begin{aligned}
&|Q_2(x(s),y(s),s) - Q_2(x^*(s),y^*(s),s)| \\
&\quad\leq \frac{1}{(1-p)}|x(s)-x^*(s)| + |x(s)^p - x^*(s)^p| + \frac{s}{2}|y(s)-y^*(s)| \\
&\quad\leq \frac{1}{(1-p)}|x(s)-x^*(s)| + p(\theta(s))^{p-1}|x(s)-x^*(s)| + \frac{\eta^*}{2}|y(s)-y^*(s)| \\
&\quad\leq \left(\frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1}\right)|x(s)-x^*(s)| + \frac{\eta^*}{2}|y(s)-y^*(s)| \\
&\quad\leq \left(\frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^*}{2}\right)v(s)
\end{aligned} \tag{116}$$

for all $s \in (0,\eta^*]$. Now, via (9), (10), (14), (105), (116) and (114), we have,

$$\begin{aligned}
v(\eta) &\leq \int_0^\eta (|Q_1(x(s),y(s),s) - Q_1(x^*(s),y^*(s),s)| \\
&\qquad + |Q_2(x(s),y(s),s) - Q_2(x^*(s),y^*(s),s)|)\,ds \\
&\leq \int_0^{\eta_\delta} K(p,\beta)(v(s))^p ds + \int_{\eta_\delta}^\eta \left(1 + \frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^*}{2}\right)v(s)ds \\
&\leq \frac{\delta}{(1-p)} + \int_{\eta_\delta}^\eta \left(1 + \frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^*}{2}\right)v(s)ds
\end{aligned} \tag{117}$$

for all $\eta \in [\eta_\delta,\eta^*]$. An application of Gronwall's Lemma [26, Corollary 6.2] to (117), gives

$$v(\eta) \leq \frac{\delta}{(1-p)} e^{\left(\eta^*\left(1+\frac{1}{(1-p)}+\left(\frac{\beta\eta^*}{2}\right)^{p-1}+\frac{\eta^*}{2}\right)\right)} \tag{118}$$

for all $\eta \in [\eta_\delta,\eta^*]$. Since $v$ is non-negative and $\eta^*$ is independent of $\delta$, it follows from (118) and (114), upon letting $\delta \to 0$, that

$$v(\eta) = 0 \quad \forall \eta \in [0,\eta^*]. \tag{119}$$

Finally, it follows from (119) and (108) that

$$(x(\eta),y(\eta)) = (x^*(\eta),y^*(\eta)) \quad \forall \eta \in [0,\eta^*],$$

as required. □

We can now state the following uniqueness result.

**Lemma 14.** *For each $(0,\beta) \in \partial\Omega_1$ then (S) with zero-value $(0,\beta)$ has exactly one solution $(x,y) : I \to \mathbb{R}^2$. This solution satisfies exactly one of the cases: (i) (with $I = [0,\eta_\beta + \epsilon_\beta]$), (ii) (with $I = [0,\eta_\beta + \epsilon_\beta]$) or (iii) (with $I = [0,\infty)$).*

*Proof.* We have established earlier that for each $(0,\beta) \in \partial\Omega_1$, then (S) with zero-value $(0,\beta)$ has at least one solution $(x,y) : I \to \mathbb{R}^2$, and that the solution satisfies one of the cases (i)-(iii). It follows from Proposition 13 that this solution is unique for $\eta \in [0,\eta^*]$, (with $\eta^*$ depending only upon $\beta$ and $p$) and, moreover, in whichever case of (i)-(iii) it falls, that $(x(\eta),y(\eta)) \notin \{(0,\lambda) : \lambda \in \mathbb{R}\}$ for any $\eta \in I \setminus [0,\eta^*]$. Repeated application of the classical uniqueness theorem [24, Chapter 1, Theorem 2.2] then completes the uniqueness result for $\eta \in I \setminus [0,\eta^*]$. □

We immediately obtain a continuous dependence result for solutions of (S) with zero-value in $\partial\Omega_1$, namely,

**Corollary 15.** *Let $(0,\beta^*) \in \partial\Omega_1$ and suppose that the unique solution to (S) with zero-value $(0,\beta^*)$, say $(x^*,y^*) : I \to \mathbb{R}$, satisfies case (i) or (ii), with $I = [0,\eta_{\beta^*} + \epsilon_{\beta^*}]$. Then, given $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $\beta > 0$ satisfying $|\beta - \beta^*| < \delta'$, the corresponding unique solution to (S) with zero-value $(0,\beta)$, say $(x,y) : I' \to \mathbb{R}$, has $I' = I$ and satisfies the corresponding case (i) or (ii), with,*

$$|x(\eta) - x^*(\eta)| + |y(\eta) - y^*(\eta)| < \epsilon' \quad \forall \eta \in I.$$

*Proof.* We first recall that (for a suitable choice of $\beta^*$) then

$$|x^*(\eta)| \le (1-p)^{\frac{1}{1-p}} + 1, \quad |y^*(\eta)| \le \beta^* + 1 \quad \forall \eta \in [0,\eta_{\beta^*} + \epsilon_{\beta^*}],$$

and, via (14), that $\mathbf{Q}(x,y,\eta)$ is continuous (and therefore bounded) on the rectangle

$$R = \left\{(x,y,\eta) : |x| \le (1-p)^{\frac{1}{1-p}} + 1, \quad |y| \le \beta^* + 1, \quad 0 \le \eta \le \eta_{\beta^*} + \epsilon_{\beta^*}\right\}.$$

The *uniqueness result* in Lemma 14 then allows for an application of the result [24, Theorem 4.3, pp. 59] which completes the proof. □

It is now convenient to introduce the three sets $E_1$, $E_2$ and $E_3$, where

$$E_1 = \{(0,\beta) \in \partial\Omega_1 \text{: the unique solution to (S) with zero-value } (0,\beta) \text{ satisfies case (i)}\},$$

with $E_2$ and $E_3$ defined similarly for cases (ii) and (iii) respectively. It follows from Lemma 14 that

$$E_i \cap E_j = \varnothing \ \text{ for } \ i,j = 1,2,3 \text{ with } i \neq j, \tag{120}$$

whilst

$$E_1 \cup E_2 \cup E_3 = \partial\Omega_1. \tag{121}$$

We now establish that $E_1$ and $E_2$ are both nonempty.

**Proposition 16.** *The set $E_1$ is non-empty and is such that $(0,\beta) \in E_1$ for each*

$$\beta > \sqrt{2\left(((1-p)^{1/(1-p)} - m_H)^2 - m_H^2\right)}, \tag{122}$$

*with $m_H$ given by* (99).

*Proof.* Let $(x,y) : I \to \mathbb{R}^2$ be the unique solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$ and $\beta$ satisfying (122). Since $(x(\eta),y(\eta)) \in \bar{\Omega}$ for all $\eta \in I'$ (where $I' = [0,\eta_\beta]$ for cases (i) and (ii), and $I' = [0,\infty)$ for case (iii)) then, via (9) and (10), we have,

$$\frac{\beta}{2} \le y(\eta) \le \beta, \quad x(\eta) \ge \frac{\beta\eta}{2} \quad \forall \eta \in [0,\bar{\eta}_\beta], \tag{123}$$

with,

$$\bar{\eta}_\beta = \begin{cases} \min\{\eta_\beta, \eta'_\beta\} & : \text{ cases (i) and (ii)} \\ \eta'_\beta & : \text{ case (iii)} \end{cases} \tag{124}$$

and

$$\eta'_\beta = \frac{2}{\beta}\left(m_H + \sqrt{m_H^2 + \frac{\beta^2}{2}}\right).$$

Now suppose case (iii) occurs, then $(x(\eta'_\beta), y(\eta'_\beta)) \in \Omega$. However,

$$x(\eta'_\beta) \geq \frac{\beta\eta'_\beta}{2} = m_H + \sqrt{m_H^2 + \frac{\beta^2}{2}} > (1-p)^{\frac{1}{1-p}}$$

via (123) and (122), and we arrive at a contradiction. We can therefore eliminate case (iii). Next suppose case (ii) occurs. It follows from $(123)_2$ and (124) that $\eta_\beta \leq \eta'_\beta$, and so $\bar{\eta}_\beta = \eta_\beta$. Thus, via $(123)_1$,

$$y(\eta_\beta) \geq \frac{\beta}{2} > 0.$$

However, in case (ii), $y(\eta_\beta) = 0$, and we arrive at a contradiction. We conclude finally that case (i) must occur, as required. □

We can also establish a similar result for $E_2$.

**Proposition 17.** *The set $E_2$ is non-empty and is such that $(0,\beta) \in E_2$ for each*

$$0 < \beta < \sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}}. \tag{125}$$

*Proof.* It follows from (16)-(21) that for $\beta$ satisfying the inequality (125), then $(0,\beta) \in D_{c^*(p)}$. It then follows from Corollary 4 that (S) with zero-value $(0,\beta)$ has a global solution which lies in $D_{c_\beta}$ for all $\eta \in (0,\infty)$ with $c_\beta = V(0,\beta) < c^*(p)$, and so the solution to (S) in $\eta \geq 0$ must satisfy case (ii). Therefore, $(0,\beta) \in E_2$, as required. □

We next establish that both $E_1$ and $E_2$ are open subsets of $\partial\Omega_1$.

**Proposition 18.** *The sets $E_1$ and $E_2$ are open subsets of $\partial\Omega_1$.*

*Proof.* We will prove the result for $E_1$. The proof for $E_2$ is similar. Let $(0,\beta^*) \in E_1$. Then, via Lemma 14, (S) with zero-value $(0,\beta^*)$ has a unique solution $(x^*, y^*) : [0, \eta_{\beta^*} + \epsilon_{\beta^*}] \to \mathbb{R}^2$, with

$$(x^*(\eta), y^*(\eta)) \in \Omega \quad \forall \eta \in (0, \eta_{\beta^*}) \tag{126}$$

and

$$(x^*(\eta_{\beta^*}), y^*(\eta_{\beta^*})) = ((1-p)^{1/(1-p)}, y_{\beta^*}) \tag{127}$$

for some $0 < y_{\beta^*} < \beta^*$, whilst

$$(x^*(\eta), y^*(\eta)) \notin \bar{\Omega} \quad \forall \eta \in (\eta_{\beta^*}, \eta_{\beta^*} + \epsilon_{\beta^*}]. \tag{128}$$

Now consider the family of open balls

$$B(x^*(\eta), y^*(\eta); \epsilon') \text{ with } \eta \in [0, \eta_{\beta^*} + \epsilon_{\beta^*}]$$

and via (126)-(128), choose $\epsilon'$ sufficiently small so that

$$B(x^*(\eta_{\beta^*} + \epsilon_{\beta^*}), y^*(\eta_{\beta^*} + \epsilon_{\beta^*}); \epsilon') \cap \bar{\Omega} = \varnothing \tag{129}$$

and

$$\bigcup_{\lambda\in[0,\eta_{\beta^*}+\epsilon_{\beta^*}]} B(x^*(\lambda),y^*(\lambda);\epsilon')\cap(\partial\Omega\setminus\partial\Omega_1)\subset\{((1-p)^{1/(1-p)},\lambda):\lambda>0\}. \tag{130}$$

It then follows from Corollary 15 that there exists $\delta'>0$ such that the corresponding unique solution to (S) with zero-value $(0,\beta)\in\partial\Omega_1$, satisfying $|\beta-\beta^*|<\delta'$, say $(x,y):[0,\eta_{\beta^*}+\epsilon_{\beta^*}]\to\mathbb{R}^2$ has

$$(x(\eta),y(\eta))\in\bigcup_{\lambda\in[0,\eta_{\beta^*}+\epsilon_{\beta^*}]} B(x^*(\lambda),y^*(\lambda);\epsilon')\quad\forall\eta\in[0,\eta_{\beta^*}+\epsilon_{\beta^*}] \tag{131}$$

Therefore, via (129)-(131), $\{(0,\beta):|\beta-\beta^*|<\delta'\}\subseteq E_1$, and so $E_1$ is an open subset of $\partial\Omega_1$, as required. □

Finally, we have

**Corollary 19.** *The set $E_3$ is a non-empty closed subset of $\partial\Omega_1$.*

*Proof.* Via Propositions 16 and 17, $E_1$ and $E_2$ are both nonempty subsets of $\partial\Omega_1$. Moreover, via (120) $E_1$ and $E_2$ are disjoint. Suppose that $E_3$ is empty, then via (121) and Proposition 18, $E_1$ and $E_2$ form an open partition of $\partial\Omega_1$. However, $\partial\Omega_1$ is a connected subset of $\mathbb{R}^2$, and we arrive at a contradiction. Hence $E_3$ must be nonempty. Finally, $E_3=\partial\Omega_1\backslash(E_1\cup E_2)$ and is therefore a closed subset of $\partial\Omega_1$. □

**Remark 3.** In Corollary 19, the existence of at least one point in $E_3$ has been established. However, it has not been established that this is the only point in $E_3$.

To conclude this section, we arrive at our main result, namely,

**Theorem 20.** *There exists a solution $(x,y):\mathbb{R}\to\mathbb{R}^2$ to (S) with zero-value $(0,\beta)\in\partial\Omega_1$, for some*

$$\sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}}\le\beta\le\sqrt{2\left(((1-p)^{1/(1-p)}-m_H)^2-m_H^2\right)},$$

*which satisfies*

$$(x(\eta),y(\eta))\to(\pm(1-p)^{1/(1-p)},0)\ \text{as}\ \eta\to\pm\infty \tag{132}$$

*and*

$$|x(\eta)|<(1-p)^{1/(1-p)},\quad 0<y(\eta)\le\beta\quad\forall\eta\in\mathbb{R}. \tag{133}$$

*Proof.* It follows directly from Corollary 19 and (iii) that there exists $(x^*,y^*):[0,\infty)\to\mathbb{R}^2$ which is a solution to (S) with zero-value $(0,\beta^*)$, such that

$$(x^*(\eta),y^*(\eta))\to((1-p)^{1/(1-p)},0)\ \text{as}\ \eta\to\infty, \tag{134}$$

$$(x^*(\eta),y^*(\eta))\in\Omega\quad\forall\eta\in(0,\infty). \tag{135}$$

It follows from (125) and (122), that

$$\sqrt{\frac{(1-p)^{1/(1-p)}}{(1+p)}}\le\beta^*\le\sqrt{2\left(((1-p)^{1/(1-p)}-m_H)^2-m_H^2\right)}.$$

Now, define the function $(x,y):\mathbb{R}\to\mathbb{R}^2$ to be

$$(x(\eta),y(\eta))=\begin{cases}(x^*(\eta),y^*(\eta)) & ;\eta\in[0,\infty)\\ (-x^*(-\eta),y^*(-\eta)) & ;\eta\in(-\infty,0).\end{cases} \tag{136}$$

It follows from (136) that $(x,y):\mathbb{R}\to\mathbb{R}^2$ is a solution to (S) with zero-value $(0,\beta^*)$, and via (94) and (iii), (since $y(\eta)$ is monotone decreasing for $\eta\in(0,\infty)$) that this solution satisfies (132) and (133). □

We conclude from Theorem 20 that the problem (S) has at least one heteroclinic connection from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ $(\eta = -\infty)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$ $(\eta = \infty)$, which we denote by $w_{\beta^*} : \mathbb{R} \to \mathbb{R}$. Here $w = w_{\beta^*}(\eta)$, $\eta \in \mathbb{R}$, has zero-value $w(0) = 0$, $w'(0) = \beta^*$ for some

$$\sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}} \leq \beta^* \leq \sqrt{2\left(((1-p)^{1/(1-p)} - m_H)^2 - m_H^2\right)},$$

and

$$|w_{\beta^*}(\eta)| < (1-p)^{1/(1-p)}, \quad 0 < w'_{\beta^*}(\eta) \leq \beta^* \quad \forall \eta \in \mathbb{R},$$

recalling also, that $w_{\beta^*}(\eta)$ is an odd function of $\eta \in \mathbb{R}$. Finally, a straightforward linearization as $|\eta| \to \infty$ establishes that,

$$w_{\beta^*}(\eta) \sim \pm(1-p)^{1/(1-p)} - \frac{A_\infty}{\eta^3} e^{-\frac{1}{4}\eta^2} \text{ as } \eta \to \pm\infty,$$

with $A_\infty$ being a globally determined constant.

*4.2. Front Solutions to [CP]*

Following Theorem 20, with $\beta = \beta^*$ we have constructed the front-like global solution $u_{\beta^*} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ to [CP], namely,

$$u_{\beta^*}(x,t) = \begin{cases} t^{\frac{1}{(1-p)}} w_{\beta^*}\left(\frac{x}{t^{1/2}}\right) & ,(x,t) \in \mathbb{R} \times (0,\infty) \\ 0 & ,(x,t) \in \mathbb{R} \times \{0\}. \end{cases} \tag{137}$$

We again observe that $u_{\beta^*}(x - x_0, t)$ is also a global solution to [CP] for any fixed $x_0 \in \mathbb{R}$. In addition, following Section 3.3, we conclude that, for any $\tau > 0$, $u^\tau_{\beta^*} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that

$$u^\tau_{\beta^*}(x,t) = \begin{cases} (t-\tau)^{\frac{1}{(1-p)}} w_{\beta^*}\left(\frac{x}{(t-\tau)^{1/2}}\right) & ,(x,t) \in \mathbb{R} \times (\tau,\infty) \\ 0 & ,(x,t) \in \mathbb{R} \times [0,\tau] \end{cases}$$

is also a front-like global solution to [CP].

## 5. Discussion

There are two questions that arise naturally from this study. The first being how one can rigorously establish the decay rate of the homoclinic solutions $w : \mathbb{R} \to \mathbb{R}$ to (S) as $\eta \to \pm\infty$, that is suggested by (87) and (88); the second being whether or not for the problem (S), there is a unique heteroclinic connection from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$ which has zero value in $\partial\Omega_1$ (Theorem 20 guarantees that there exists at least one connection).

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